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Stability Analysis on Earth Observing System AM-1 Spacecraft Attitude Determination

Xipu Li*
Lockheed Martin Missiles and Space, Princeton, New Jersey 08543
Jeffrey B. Boka†
Lockheed Martin Missiles and Space, King of Prussia, Pennsylvania 19046
and
Arthur J. Throckmorton‡
Lockheed Martin Missiles and Space, Sunnyvale, California 94089

Nomenclature

$\boldsymbol{S}$ = span$\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$, linear subspace spanned by vectors $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$
$\|\boldsymbol{v}\|$ = 2-norm of a vector $\boldsymbol{v}$ defined as $\|\boldsymbol{v}\| := \sqrt{(v_x^2 + v_y^2 + v_z^2)}$
$[\boldsymbol{v}]_s$ = skew symmetric matrix of a vector

$$\boldsymbol{v} = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$$

defined as

$$[\boldsymbol{v}]_s = \begin{bmatrix} 0 & -v_z & v_y \\ v_z & 0 & -v_x \\ -v_y & v_x & 0 \end{bmatrix}$$

$\mathbf{0}_3$ = 3×3 zero matrix
$\mathbf{1}_3$ = 3×3 identity matrix

I. Introduction

THE Earth observing system (EOS) spacecraft series is the cornerstone of NASA's mission to planet Earth. The first spacecraft (SC), EOS AM-1, is currently being developed by Lockheed Martin Missiles and Space and is scheduled for launch in 1998.

The mission of the EOS AM-1 SC requires that the attitude control system have a highly accurate Earth-pointing controllability, which in turn demands that the SC attitude determination system be able to provide an accurate attitude of the vehicle. The primary sensors used for EOS AM-1 attitude determination include an inertia reference unit containing three sets of two-axis gyroscopes and a set of optical sensors including two solid-state star trackers. The attitude determination process is implemented with the Kalman filter algorithm; a sixth-order linear time-varying state estimator is used to reconstruct the three-axis attitude determination errors and the three-axis gyro bias. The Kalman filter algorithm has been successfully applied to the area of SC attitude determination for more than two decades. A complete historical survey can be found in Ref. 1. As is known, the optimality of the Kalman filter, in general, does not necessarily imply its convergence or stability.[2] In many design cases, the stability of the Kalman filter is due to the nature of the system, namely, observability; the stability is usually verified by simulations without further theoretical justifications. Hence, the obvious question: is the Kalman filter algorithm used in the gyro/star-tracker attitude determination system always stable? And if not, what is the condition for the stability?

This Note presents an analytical investigation on the stability of the stellar-inertial type of attitude determination system. The approach is based on the observability analysis for a time-varying system. The structure of the unobservable subspace is investigated along with its physical explanations and implications. The analysis indicates that, if a few star measurements are available around an orbit, the overall system is completely observable. Together with a discussion on the system controllability, the system stability is proven by employing the Kalman stability theorem.

II. Mathematical Model

A. Derivation of the State Equation

Let $\boldsymbol{w}$ represent the true SC body rate vector in the $\boldsymbol{SC}_t$ frame, that is, the true SC body frame (attitude) with respect to the inertial reference frame (***ECI*** frame), and $\boldsymbol{w}_m$ the gyro measured body rate vector in the $\boldsymbol{SC}_k$ frame, that is, the estimated SC body frame (attitude) with respect to the ***ECI*** frame. The measured body rate vector includes white noise $\boldsymbol{n}_1$ and the gyro rate bias $\boldsymbol{w}_{\text{bias}}$, which is modeled as random walk noise (integration of the white noise vector $\boldsymbol{n}_2$) (Ref. 3),

$$\boldsymbol{w}_m = \boldsymbol{w} + \boldsymbol{w}_{\text{bias}} - \boldsymbol{n}_1 \tag{1}$$

$$\dot{\boldsymbol{w}}_{\text{bias}} = \boldsymbol{n}_2 \tag{2}$$

Because the true SC attitude $\boldsymbol{SC}_t$ is rotating about the true rate vector $\boldsymbol{w}$ and the estimated attitude $\boldsymbol{SC}_k$ is rotating about the measured rate vector $\boldsymbol{w}_m$, the resulting attitude determination error $\boldsymbol{e}$ from $\boldsymbol{SC}_k$ to $\boldsymbol{SC}_t$ satisfies the following equation[4]:

$$\dot{\boldsymbol{e}} = \boldsymbol{w}_e = \boldsymbol{T}_e^{-1}\boldsymbol{w} - \boldsymbol{w}_m \tag{3}$$

where $\boldsymbol{T}_e$ is the transformation from the estimated attitude $\boldsymbol{SC}_k$ to the true attitude $\boldsymbol{SC}_t$ and $\boldsymbol{w}_e$ is the rate vector of the $\boldsymbol{SC}_t$ frame relative to the $\boldsymbol{SC}_k$ frame. By applying the small angle approximation

$$\boldsymbol{T}_e^{-1} = (\mathbf{1}_3 + [\boldsymbol{e}]_s) \tag{4}$$

the state equation can be obtained as

$$\begin{bmatrix} \dot{\boldsymbol{e}} \\ \dot{\boldsymbol{w}}_{\text{bias}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{W} & -\mathbf{1}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 \end{bmatrix} \begin{bmatrix} \boldsymbol{e} \\ \boldsymbol{w}_{\text{bias}} \end{bmatrix} + \begin{bmatrix} \boldsymbol{n}_1 \\ \boldsymbol{n}_2 \end{bmatrix} \tag{5}$$

where $\boldsymbol{W} = -[\boldsymbol{w}]_s$.

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*Staff Systems Engineer, Guidance, Navigation, and Control, M/S WC-2. Member AIAA.
†Staff Systems Engineer, Guidance, Navigation, and Control, M/S U2101.
‡Director, Flight Controls Product Center, Organization E5-01, Building 104. Senior Member AIAA.

During the normal mode of the EOS AM-1 mission, the SC will be controlled to rotate about its pitch axis with a constant rate $\boldsymbol{w}_0$. Because the true body rate vector $\boldsymbol{w}$ is not available for Eq. (5), $\boldsymbol{w}_0$ is used instead, which makes the $\boldsymbol{A}$ matrix a constant matrix.

B. Derivation of the Output Equation

The system output variable $\boldsymbol{y}(t)$ is defined as the star measurement residual

$$\boldsymbol{y}(t) = [\boldsymbol{s}_m(t) + \boldsymbol{n}_m] - \boldsymbol{s}_c(t) \tag{6}$$

where $[\boldsymbol{s}_m(t) + \boldsymbol{n}_m]$ is a measured star vector from a star tracker, $\boldsymbol{s}_c(t)$ is the corresponding star-catalog star vector in the $\boldsymbol{SC}_k$ frame, and $\boldsymbol{n}_m$ is measurement noise. The two star vectors can be related by the following equations:

$$\boldsymbol{s}_c(t) = \boldsymbol{T}_k\boldsymbol{s}_{ci}(t), \qquad \boldsymbol{s}_m(t) = \boldsymbol{T}_t\boldsymbol{s}_{ci}(t), \qquad \boldsymbol{T}_k = \boldsymbol{T}_e^{-1}\boldsymbol{T}_t$$

where $\boldsymbol{s}_{ci}(t)$ is the corresponding star-catalog star vector in the $\boldsymbol{ECI}$ frame and $\boldsymbol{T}_t$ and $\boldsymbol{T}_k$ are the transformations from the $\boldsymbol{ECI}$ frame to $\boldsymbol{SC}_t$ and $\boldsymbol{SC}_k$ frames, respectively. With Eq. (4), the output equation can be written as

$$\boldsymbol{y}(t) = \boldsymbol{H}(t)\begin{bmatrix}\boldsymbol{e}\\ \boldsymbol{w}_{\text{bias}}\end{bmatrix} + \boldsymbol{n}_m \tag{7a}$$

with

$$\boldsymbol{H}(t) = \begin{bmatrix}[\boldsymbol{s}_m(t)]_s & \boldsymbol{0}_3\end{bmatrix} \tag{7b}$$

C. Solution to the State Equation

The solution to the system [Eq. (5)] can be derived as $\boldsymbol{x}(t) = \boldsymbol{\Phi}(t)\boldsymbol{x}(t_0)$, where

$$\boldsymbol{\Phi}(t) = \begin{bmatrix}\boldsymbol{\Phi}_1(t) & \boldsymbol{\Phi}_2(t)\\ \boldsymbol{0}_3 & \boldsymbol{1}_3\end{bmatrix} \tag{8a}$$

$$\boldsymbol{\Phi}_1(t) = \boldsymbol{1}_3 + \alpha_1\boldsymbol{W} + \alpha_2\boldsymbol{W}^2 \tag{8b}$$

$$\boldsymbol{\Phi}_2(t) = \beta_0\boldsymbol{1}_3 + \beta_1\boldsymbol{W} + \beta_2\boldsymbol{W}^2 \tag{8c}$$

$$\alpha_1 = \frac{\sin[\|\boldsymbol{w}\|(t-t_0)]}{\|\boldsymbol{w}\|}, \qquad \alpha_2 = \frac{1-\cos[\|\boldsymbol{w}\|(t-t_0)]}{\|\boldsymbol{w}\|^2}$$

$$\beta_0 = -(t-t_0), \qquad \beta_1 = -\alpha_2,$$

$$\beta_2 = -\frac{\|\boldsymbol{w}\|(t-t_0) - \sin[\|\boldsymbol{w}\|(t-t_0)]}{\|\boldsymbol{w}\|^3}$$

The following summarizes some useful properties of the transition matrix $\boldsymbol{\Phi}(t)$.

Result 1:

1) $\boldsymbol{\Phi}_1(t)$ and $\boldsymbol{\Phi}(t)$ are full rank for $t \geq t_0$; $\boldsymbol{\Phi}_2(t)$ is full normal rank for $t \geq t_0$.

2) The SC rate vector $\boldsymbol{w}$ is a time-invariant eigenvector of both $\boldsymbol{\Phi}_1(t)$ and $\boldsymbol{\Phi}_2(t)$.

Proof: Part 1 can be proven by solving eigenvalues of $\boldsymbol{\Phi}(t)$. Because the eigenvalues of $\boldsymbol{W}$ are 0 and $\pm j\|\boldsymbol{w}\|$, it is straightforward to show that the eigenvalues of $\boldsymbol{\Phi}(t)$ are

$$1,\ 1,\ 1,\ 1,\ \cos[\|\boldsymbol{w}\|(t-t_0)] \pm j\sin[\|\boldsymbol{w}\|(t-t_0)]$$

Hence, both $\boldsymbol{\Phi}_1(t)$ and $\boldsymbol{\Phi}(t)$ are nonsingular for any t. By using the same technique, the eigenvalues of $\boldsymbol{\Phi}_2(t)$ can be obtained as

$$-(t-t_0), \qquad -\frac{1}{\|\boldsymbol{w}\|}(\sin[\|\boldsymbol{w}\|(t-t_0)] \pm j\{1-\cos[\|\boldsymbol{w}\|(t-t_0)]\})$$

which means that $\boldsymbol{\Phi}_2(t)$ is full rank for any t, except for $t = t_0 + (2k\pi/\|\boldsymbol{w}\|)$, $k = 0, 1, 2, \ldots$.

Part 2 of the result can be proven by simply using the fact that $\boldsymbol{w}$ is an eigenvector of $\boldsymbol{W}$ corresponding to its 0 eigenvalue.

To complete this section, we include the kinematic equation of the star vector $\boldsymbol{s}_m(t)$ in the body frame,

$$\dot{\boldsymbol{s}}_m(t) = -[\boldsymbol{w}]_s\boldsymbol{s}_m(t) = \boldsymbol{W}\boldsymbol{s}_m(t) \tag{9a}$$

$$\boldsymbol{s}_m(t) = \boldsymbol{\Phi}_1(t)\boldsymbol{s}_m(t_0) \tag{9b}$$

III. Observability, Controllability, and Stability of the System

The following definitions on linear system observability are standard and can be found in many textbooks.[5]

Definition 1: A linear time-varying system $[\boldsymbol{A}, \boldsymbol{H}(t)]$ is completely observable on $[t_0, +\infty]$ if and only if

$$\boldsymbol{y}(t) = \boldsymbol{H}(t)\boldsymbol{x}(t) = \boldsymbol{H}(t)\boldsymbol{\Phi}(t)\boldsymbol{x}(t_0) \equiv \boldsymbol{0}, \qquad t \geq t_0$$

implies

$$\boldsymbol{x}_0 = \boldsymbol{x}(t_0) = \boldsymbol{0}$$

Definition 2: The unobservable subspace of the system $\boldsymbol{X}_{uo}$ consists of all of the vectors $\boldsymbol{x}_0$ such that

$$\boldsymbol{y}(t) = \boldsymbol{H}(t)\boldsymbol{\Phi}(t)\boldsymbol{x}_0 \equiv \boldsymbol{0} \qquad \text{for all} \qquad t \geq t_0$$

With these definitions, we present the following result on the observability of the attitude determination system.

Result 2: The system $[\boldsymbol{A}, \boldsymbol{H}(t)]$ defined by Eqs. (5) and (7) is not completely observable with only one star measurement vector $\boldsymbol{s}_m(t)$. Moreover, 1) its normal unobservable subspace is one dimensional and spanned by

$$\boldsymbol{X}_{\text{uo}} = \text{span}\left(\begin{bmatrix}\boldsymbol{s}_m(t_0)\\ \boldsymbol{0}\end{bmatrix}\right) \tag{10}$$

and 2) a singularity case occurs when $\boldsymbol{s}_m(t)$ is aligned with $\boldsymbol{w}$, the SC rotating vector. In this case, the unobservable subspace is two dimensional and spanned by constant vectors

$$\boldsymbol{X}_{\text{uo}} = \text{span}\left(\begin{bmatrix}\boldsymbol{w}\\ \boldsymbol{0}\end{bmatrix}, \begin{bmatrix}\boldsymbol{0}\\ \boldsymbol{w}\end{bmatrix}\right) \tag{11}$$

Proof: Based on the derivations in Sec. II, we have

$$\begin{aligned}\boldsymbol{y}(t) &= \begin{bmatrix}[\boldsymbol{s}_m(t)]_s & \boldsymbol{0}_3\end{bmatrix}\begin{bmatrix}\boldsymbol{\Phi}_1(t) & \boldsymbol{\Phi}_2(t)\\ \boldsymbol{0}_3 & \boldsymbol{1}_3\end{bmatrix}\begin{bmatrix}\boldsymbol{x}_{01}\\ \boldsymbol{x}_{02}\end{bmatrix}\\ &= [\boldsymbol{s}_m(t)]_s[\boldsymbol{\Phi}_1(t)\boldsymbol{x}_{01} + \boldsymbol{\Phi}_2(t)\boldsymbol{x}_{02}] \qquad \text{for all} \qquad t \geq t_0\end{aligned} \tag{12}$$

By using Eq. (9b) and the fact that the null space of $[\boldsymbol{s}_m(t)]_s$ is spanned by $\boldsymbol{s}_m(t)$, it is easy to see that any initial state vector in the form of

$$\boldsymbol{x}_0 = k\begin{bmatrix}\boldsymbol{s}_m(t_0)\\ \boldsymbol{0}\end{bmatrix} \tag{13}$$

will cause the output $\boldsymbol{y}(t) \equiv \boldsymbol{0}$ for all $t \geq t_0$, where k is an arbitrary constant. Therefore,

$$k\begin{bmatrix}\boldsymbol{s}_m(t_0)\\ \boldsymbol{0}\end{bmatrix} \in \boldsymbol{X}_{\text{uo}} \tag{14}$$

On the other hand, we want to prove that any vector $\boldsymbol{x}_0$ in $\boldsymbol{X}_{\text{uo}}$ is in the form of Eq. (13). To this end, we start with an arbitrary vector $\boldsymbol{x}_0 \in \boldsymbol{X}_{\text{uo}}$. By the definition of $\boldsymbol{X}_{\text{uo}}$,

$$\boldsymbol{y}(t) = \boldsymbol{H}(t)\boldsymbol{\Phi}(t)\boldsymbol{x}_0 \equiv \boldsymbol{0} \qquad \text{for all} \qquad t \geq t_0 \tag{15}$$

Because this equation is also true at $t = t_0$, we have

$$y(t_0) = H(t_0)\Phi(t_0)x_0 = [s_m(t_0)]_s x_{01} = 0 \tag{16}$$

which implies

$$x_{01} = ks_m(t_0) \tag{17}$$

Furthermore, with Eq. (17), Eq. (15) can be reduced to

$$y(t) = [s_m(t)]_s \Phi_2(t)x_{02} \equiv 0 \qquad \text{for all} \qquad t \geq t_0 \tag{18}$$

Note that, for a nonzero x_{02}, Eq. (18) is true if and only if $\Phi_2(t)x_{02} \equiv 0$ or $\Phi_2(t)x_{02}$ is parallel with $s_m(t)$. However, because $\Phi_2(t)x_{02} \neq 0$ for all $t \geq t_0$ (Result 1) and $\Phi_2(t)x_{02} \neq ks_m(t)$, in general (see the next paragraph), Eq. (18) is true if and only if $x_{02} = 0$, which completes the proof of the first part of Result 2.

It is straightforward to see that $\Phi_2(t)x_{02} \neq ks_m(t)$, in general, because $\Phi_1(t) \neq \Phi_2(t)$. However, the singular case occurs when the initial star vector $s_m(t_0)$ is parallel with the SC rotating vector w. In this case,

$$s_m(t) = \frac{w}{\|w\|} \qquad \text{for all} \qquad t \geq t_0 \tag{19}$$

Therefore, for a nonzero x_{02}, Eq. (18) is true if and only if x_{02} is chosen such that $\Phi_2(t)x_{02}$ is parallel with w for all $t \geq t_0$. Because w is the only constant eigenvector of $\Phi_2(t)$ (Result 1), the only solution is $x_{02} = kw$. Hence, we complete the proof of the second part of Result 2.

Result 2 has the following implications. In the normal case, the only unobservable variable of the system is the component of the attitude determination error vector $e(t)$ in the direction of the measured star vector $s_m(t)$. This portion of $e(t)$ cannot be determined from $y(t)$ and, therefore, is uncorrectable. If the initial vector $e(t_0)$ is parallel with $s_m(t_0)$, then the entire motion of $e(t)$ will be confined to be parallel with $s_m(t)$, being unobservable all of the time. The trajectory of the unobservable $e(t)$ is sinusoidal in the body frame and a fixed vector in the inertial frame. The gyro bias vector w_{bias} is completely observable in this case. The singular case occurs when $s_m(t)$ is parallel with w. The components of both $e(t)$ and w_{bias} in the direction of $s_m(t)$ are unobservable. Because the trajectory of $s_m(t)$ is a body-fixed vector instead of a cone as is in the normal case, the motions of these unobservable variables are constant in both the body frame and the inertial frame.

The following paragraphs will address the fact that two distinct inertial vectors are needed to completely determine the SC attitude.

Result 3: The SC attitude determination error can be fully determined if a few (at least two) star measurements around an orbit are available.

Proof: It suffices to prove the case when a new star measurement is available to update the initial measurement. In this case, the measurement vector can be expressed as a piecewise continuous time function for $t \geq t_0$:

$$s_m(t) = \begin{cases} s_{m1}(t) & t_0 \leq t < t_1 \\ s_{m2}(t) & t_1 \leq t \end{cases} \tag{20}$$

where t_1 is the time instant when the second measurement vector $s_{m2}(t)$ is acquired. Because $s_{m1}(t)$ and $s_{m2}(t)$ are linearly independent, it is straightforward to show that the null space of $[s_m(t)]_s$ is zero dimensional for $t \geq t_0$. Therefore, there does not exist a nonzero x_0 satisfying Eq. (15), which means the system is completely observable.

Result 3 has a clear physical meaning: if two or more linearly independent star observations around an orbit are available, the three-dimensional attitude determination error and three-dimensional gyro bias information can be completely extracted from the star measurement residual $y(t)$. However, in practice, more stars (more frequent measurement updates) are needed to keep a small estimation error. Without frequent updates, the estimation error grows due to gyro measurement errors, attitude propagation errors, system noises, and external disturbance.

The derivation of the system controllability is quite straightforward. The controllability here is referred to the time-invariant pair (A, D), where D is from the Cholesky decomposition of Q, the intensity matrix of system white noise vector [Eq. (5)],

$$Q = D'D \tag{21}$$

Because Q is a full-rank constant matrix, the matrix D is square, constant, and full rank. Therefore, it is trivial to show that the pair (A, D) is completely controllable.

The stability of the state estimator involves the issue of designing an estimator gain matrix $K(t)$ such that $[A + K(t)H(t)]$ is stable. This is a necessary and sufficient condition such that $\bar{x}(t) \to x(t)$ as $t \to \infty$, where $\bar{x}(t)$ is the estimation of $x(t)$. The optimality of the state estimator involves the issue of designing a $K(t)$ such that the estimation error is minimized. For a time-varying system, Kalman proved that the optimal estimator gain matrix can be solved from the following differential Riccati equation:

$$AP(t) + P(t)A' - P(t)H'(t)R^{-1}H(t)P(t) + Q = \dot{P}(t) \tag{22a}$$

$$K(t) = P(t)H'(t)R^{-1} \tag{22b}$$

where R is the intensity matrix of measurement noise [Eq. (7)]. However, the $K(t)$ obtained from Eq. (22) may not be necessarily stabilizing. The following Kalman theorem provides a sufficient condition for the $K(t)$ to be stabilizing.

Theorem 1 (Ref. 6): The state estimator is exponentially stable if 1) the entries of A, $H(t)$, Q, R^{-1} are bounded; 2) the pair $[A, H(t)]$ is completely observable; and 3) the pair (A, D) is completely controllable.

Because we have already proven these conditions, the stability of the attitude determination system follows immediately.

IV. Conclusions

Inasmuch as the Kalman filter algorithm may not necessarily converge for any system, a stability analysis of the gyro/star-tracker-type attitude determination system is necessary. This Note presented mathematical proofs on system controllability and observability that are directly related to the existence of a stable solution. The analysis indicates that the system is inherently time varying and completely controllable but not completely instantaneously observable; the unobservable subspace is one dimensional (two dimensional in a singular case). However, under the assumption that a few star measurements are available around an orbit, the overall system becomes completely observable. These conditions are then used to satisfy the Kalman stability theorem, and the stability of the system is proven.

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